STABILITY OF HOMOGENEOUS FLUID MOVING WITH FREE CONVECTION IN HORIZONTAL SLOT UNDER VARIOUS HEAT-TRANSFER CONDITIONS AT THE BOUNDING SURFACES

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A monotonically increasing relationship is shown to exist between the critical value of the parameter R_c and Bi. Numerical results are given for the R_c–Bi relationship with boundary conditions of the third kind, and results for R_c as a function of λ/λ_1 for boundary conditions of the fourth kind.

The appearance of free convection in a thin layer of a homogeneous incompressible liquid between unbounded horizontal plates heated to different temperatures is a unique phenomenon [1, 2].

Theoretical calculations performed on the assumption that the liquid temperature at the boundary equals the plate temperature show that convection always occurs when $R > 1708$ [4].

Here we consider such a problem for the case in which there is a certain temperature drop at the boundary (where, for example, there is a thin layer of material with poor thermal conductivity between the liquid and the plate, which is maintained at a constant temperature).

We allow for this temperature drop by specifying the boundary conditions of the third kind in the form

$$-\lambda \frac{\partial T}{\partial z} = \alpha_T (T - T_1) \tag{1}$$

at the boundary.

The case in which the liquid temperature at the boundary equals the plate temperature (condition of the first kind) represents a limiting case for our solution (when $\mathrm{Bi} \to \infty$).

We take the vertical axis as the z axis, and locate the origin at the center of the plane, so that the equations of the upper and lower plates will be $z = l/2$, $z = -l/2$.

The temperature T_0 of the undisturbed liquid that satisfies the equation

$$\frac{d^2T_0}{dz} = 0 \tag{2}$$

and the boundary conditions

$$-\lambda \frac{dT_0}{dz} = \alpha_T (T_1 - T_0) \quad \text{for} \quad z = -\frac{l}{2} \tag{3}$$

is represented as

$$T_0 = \frac{T_1 + T_2}{2} - \frac{\Delta T \alpha_0}{l} z. \tag{4}$$

With allowance for (4), we write the equation system for small disturbances [4, 5] in T, w in the form

$$\left(\frac{\partial}{\partial t} - a\nabla^2\right) T = \frac{\alpha_0 \Delta T}{l} w,$$

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$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\nabla^2 w = g\beta\nabla_1^2 T \tag{5}$$

with boundary conditions

$$-\lambda\frac{\partial T}{\partial z} = \pm\alpha_t T, \quad z = \pm\frac{l}{2}, \tag{6}$$

$$w = 0, \quad \frac{\partial w}{\partial z} = 0, \quad z = \pm\frac{l}{2}; \tag{7}$$

the second equation of (7) follows from the equation of continuity, since the velocity components along the x and y axes equal zero at a solid boundary.

We introduce the dimensionless coordinates and time x/l, y/l, z/l, at/l^2, and use the initial symbols x, y, z, t for them.

If in (5) we separate variables,

$$T = \alpha_0 \Delta T f(x, y)\,\theta(z)\exp(\sigma t), \quad w = \frac{a}{l} f(x, y)\,\omega(z)\exp(\sigma t), \tag{8}$$

we obtain the system

$$\begin{gathered}\nabla_1^2 f + M^2 f = 0,\\ \left(\frac{\sigma}{\mathrm{Pr}} - D\right)D\omega = -RM^2\theta,\\ (\sigma - D)\,\theta = \omega.\end{gathered} \tag{9}$$

For the variables θ, ω we write the conditions (6), (7) in the form

$$-\frac{d\theta}{dz} = \pm\,\mathrm{Bi}\,\theta \quad \text{for } z = \pm\frac{1}{2}, \tag{10}$$

$$\omega = 0, \quad \frac{d\omega}{dz} = 0 \quad \text{for } z = \pm\frac{1}{2}. \tag{11}$$

It was shown in [4] that under the zero boundary conditions for θ and the condition (11) for ω, the threshold of instability (appearance of stationary convective motion) must be determined by the equation $\sigma = 0$. (This proof is given in a short form in [3], p. 138.) This statement can also be proven for boundary condition (10). The sole peculiarity is the appearance of the additional term $I_4 = \mathrm{Bi}(|\theta(-1/2)|^2 + |\theta(1/2)|^2)$, which is positive for real Bi. We omit the proof.

We shall show that for the system considered, the critical parameter value R_c is an increasing monotonic function of the parameter Bi.

Let us assume that the system is close to the critical state. Then the parameter R is positive, while $\sigma_i = 0$. As a consequence, in the problem (9) with boundary condition (10), (11) we need only consider real values of σ (close to zero). It is not difficult to see that the problem of finding the eigenvalues and eigenfunctions is associated with the variational problem of minimizing the functional [6]

$$I(\omega, \theta, \mathrm{Bi}) = \int_{-1/2}^{1/2}\left\{\left(\frac{d^2\omega}{dz^2} + M^2\omega\right)^2 + RM^2\left[\left(\frac{d\theta}{dz}\right)^2 - 2\theta\omega\right]\right\}dz + RM^2\,\mathrm{Bi}\left[\theta^2\left(\frac{1}{2}\right) + \theta^2\left(-\frac{1}{2}\right)\right] \tag{12}$$

for a constant value of

$$H(\omega, \theta) = \int_{-1/2}^{1/2}\left[\left(\frac{d\omega}{dz}\right)^2 + RM^2\theta^2\right]dz \tag{13}$$

or under boundary conditions (11) for ω. The natural boundary conditions are imposed on the function θ. If the functions θ, ω actually minimize (12) under condition (13), then the maximum eigenvalue σ equals

$$\sigma = -\frac{I(\omega, \theta, \mathrm{Bi})}{H(\omega, \theta)}, \tag{14}$$

since when we vary (14) under the condition (11) we obtain (9) and the condition (10). From the variational principle formulated and the definition of R_c it follows that R_c is a monotonically increasing function of Bi.

TABLE 1. Values of the First Root ξ_1 of Eq. (19)

M \ Bi	0	0,01	0,1	0,25	0,5	1	5	10	∞
0,5	1,250	1,253	1,324	1,368	1,480	1,610	1,897	1,971	2,073
1	1,424	1,429	1,474	1,509	1,581	1,677	1,903	1,969	2,063
2	1,661	1,667	1,680	1,700	1,723	1,770	1,907	1,957	2,035
3	1,751	1,753	1,760	1,770	1,785	1,809	1,899	1,935	1,997
4	1,770	1,771	1,775	1,780	1,790	1,805	1,861	1,891	1,945

The minimum of R_c occurs at Bi = 0, i.e., for a boundary condition of the second kind. (We note that the value Bi = 0 for θ in (10) does not mean that the system is heat-insulated. This is equivalent to specifying a constant heat flux at the boundary.) The maximum of R_c occurs when Bi $\to \infty$. It can be shown that when Bi $\to \infty$, the maximum eigenvalue σ (14) approaches the value σ_I obtained under the zero boundary conditions for the disturbance θ. As a consequence, when Bi $\to \infty$, the limit of R_c is the value $(R_c)_I$ for a boundary condition of the first kind.

If the plates bounding the liquid are fairly thick, we will have none of the above boundary conditions for θ. In this case, the solution of (9) in the liquid must be coordinated at the boundary with the solution to the heat-conduction equation for the plate (condition of the fourth kind). If the plate is taken to be infinitely thick, the coordination conditions (requiring that the temperatures and fluxes be equal at the boundary) are easily reduced to a condition of the third kind (10), in which Bi has the following form:

$$\mathrm{Bi} = \frac{\lambda_1}{\lambda} \sqrt{\frac{a}{a_1}\sigma + M^2}, \tag{15}$$

where the subscript 1 refers to the plate.

In this case, the parameter R is written as

$$R = \frac{g\beta l^4 q}{a\nu\lambda}, \tag{16}$$

where the flux q is taken to be given.

In like manner, we can show that for a condition of the third kind the critical state of the system for (10), (15) occurs at $\sigma = 0$ and that the critical value of the parameter R_c is a monotonically increasing function of Bi; it must be remembered that Bi depends on M and σ.

Let us look at the determination of the critical values of R_c. When $\sigma = 0$, system (9) takes the form

$$\begin{gathered} D^2\omega = RM^2\theta, \\ D\theta = -\omega \end{gathered} \tag{17}$$

with boundary conditions (10), (11) for constant Bi in the first case or for

$$\mathrm{Bi} = \frac{\lambda_1}{\lambda} M \tag{18}$$

in the second.

The solution of the system (18) with boundary conditions (10), (11) is analogous to the solution with zero boundary conditions given in [4]. Thus we shall not give it in its entirety, but shall only write the transcendental equation for determining the roots, in the notation of [4], introducing additional symbols for the quantities required in our case.

This equation has the form

$$-\operatorname{tg}\xi = \frac{\mathrm{Bi}\,(\gamma\,\mathrm{sh}\,2\alpha\xi + \delta\sin 2\beta\xi) + 2\varkappa\xi\,(\mathrm{ch}\,2\alpha\xi - \cos 2\beta\xi)}{\mathrm{Bi}\,(\mathrm{ch}\,2\alpha\xi + \cos 2\beta\xi) + 2\xi\,(\gamma_1\,\mathrm{sh}\,2\alpha\xi - \delta_1\sin 2\beta\xi)}, \tag{19}$$

where α, β, γ, and δ are given in [4] (54),

$$\begin{gathered} \varkappa = \frac{A^2 + B^2}{\Lambda - 1}, \quad \gamma_1 = \frac{A - \sqrt{3}B}{\sqrt{\Lambda - 1}}, \quad \delta_1 = \frac{\sqrt{3}A + B}{\sqrt{\Lambda - 1}}, \\ \xi = \frac{M}{2}\sqrt{\Lambda - 1}, \quad M^4\Lambda^3 = R. \end{gathered} \tag{20}$$

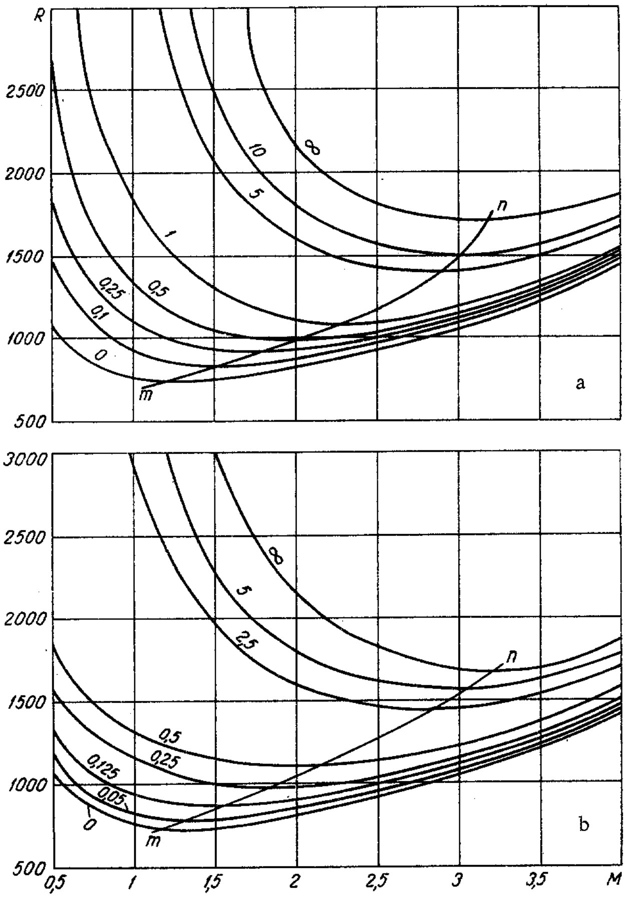

Fig. 1. Parameter R as a function of M for various Bi (boundary conditions of the first kind) (a) and λ_1/λ (boundary conditions of the fourth kind) (b); mn) line representing critical values R_c and corresponding values M_c.

The problem consists of using (19) to determine the values of the roots $\xi_j = \xi_j(\text{Bi}, M)$ as a function of the parameters Bi, M, and then finding the minimum of R with respect to M for a specified value of Bi. By definition (20),

$$R_c = \min_M R = \min_M \left[M^4 \left(1 + \frac{4\xi_j(M, \text{Bi})}{M^2} \right)^3 \right]. \tag{21}$$

It was established in [4] that under a boundary condition of the first kind, the minimum of (21) occurs for the first root of (19) when Bi $\rightarrow \infty$. From what we have proven above, under a boundary condition of the third kind, R_c increases monotonically with Bi. Thus we might expect that for finite Bi, including Bi = 0, a minimum will also occur for the first root of (19). To demonstrate this by direct computation, we first determined the values of the first three roots of (19) roughly (by a graphical method) for several values of Bi and M. The analysis showed that for finite Bi, there actually is a minimum of R for the first root; the value M_c corresponding to the critical value R_c decreases with Bi. The values of the first root of (19) were therefore determined quite accurately for a large number of Bi and M points (Table 1). Figure 1a shows R as a function of M for various Bi, on the basis of the tabulated computed values.

For boundary conditions of the fourth kind, in Eq. (19), Bi depends on M in accordance with (18). Thus for the given case, the table of roots of (19) was not recomputed, but was obtained for several values of λ_1/λ and M by quadratic interpolation at appropriate parts of the table. Figure 1b shows the calculated results for R. As we see, with increasing heat-insulation, the instability of the liquid rises. As Bi increases, the cell dimension ($\sim 1/M$) decreases.

The results of a similar analysis for an inhomogeneous fluid (binary gas mixture) will be published later.

NOTATION

Gr	is the Grashof number;
Pr	is the Prandtl number;
$\Delta T = T_1 - T_2$	is the difference in the temperatures of the upper and lower plates;
λ, λ_1	are the respective thermal conductivities of liquid and plate;
α_t	is the heat-transfer coefficient;
l	is the plate separation;
T_0	is the temperature of the undisturbed liquid;
T	is the magnitude of the temperature disturbance;
w	is the z axis velocity component;
g	is the free-fall acceleration;
$\beta = -(1/\rho)/(\partial\rho/\partial T)_p$	is the coefficient of volume expansion of the liquid;
α, α_1	is the thermal-diffusivity coefficient for the liquid and plate, respectively;
M^2	is the separation-of-variables constant;
θ	is the dimensionless value of the temperature disturbance;
ω	is the dimensionless z axis velocity component;

$\nabla^2 = \nabla_1^2 + (\partial^2/\partial z^2)$;
$\nabla_1^2 = (\partial^2/\partial x^2) + (\partial^2/\partial y^2)$;
$D = d^2/dz^2 - M^2$;
$R = g\beta l^3 \alpha_0 \Delta T/a\nu$;
$\alpha_0 = Bi/(2 + Bi)$.

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